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Killing spinors on Lorentzian manifolds

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Abstract

The aim of this paper is to describe some results concerning the geometry of Lorentzian manifolds admitting Killing spinors. We prove that there are imaginary Killing spinors on simply connected Lorentzian Einstein–Sasaki manifolds. In the Riemannian case, an odd-dimensional complete simply connected manifold (of dimension $n \neq 7$) is Einstein–Sasaki if and only if it admits a non-trivial Killing spinor to $\lambda = \pm\frac{1}{2}$. The analogous result does not hold in the Lorentzian case. We give an example of a non-Einstein Lorentzian manifold admitting an imaginary Killing spinor. A Lorentzian manifold admitting a real Killing spinor is at least locally a codimension one warped product with a special warping function. The fiber of the warped product is either a Riemannian manifold with a real or imaginary Killing spinor or with a parallel spinor, or it again is a Lorentzian manifold with a real Killing spinor. Conversely, all warped products of that form admit real Killing spinors.

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1. Introduction

In this paper, we use the notation $M^{n,k}$ for a semi-Riemannian manifold of dimension n and index k. The spinor bundle of a semi-Riemannian spin manifold $M^{n,k}$ is denoted by S, the non-degenerate scalar product on the fibers by $\langle\cdot\rangle$, the Clifford multiplication by $\cdot$ and the spinor connection by ∇. For the spin geometric notation in the Lorentzian case, see [4], a comprehensive treatment of the Riemannian case can be found in [7,10].

A *Killing spinor* on a semi-Riemannian spin manifold $(M^{n,k}, g)$ is a spinor $\varphi \in \Gamma(S)$, solving

$$\nabla^S_X \varphi = \lambda X \cdot \varphi$$

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for all $X \in TM$ and a *Killing number* $\lambda \in \mathbb{C}$. The space of solutions of the so-called *Killing equation* for $\lambda \in \mathbb{C}$ is denoted by $\mathcal{K}_\lambda(M^{n,k}, g)$.

Killing spinors were first used in mathematical physics, e.g. in supergravity theories (see [7] for detailed references on the physical literature). Later on, Killing spinors naturally appeared in Riemannian geometry, when Friedrich proved that on a compact Riemannian spin manifold (M^n, g) with positive scalar curvature, Killing spinors are exactly the eigenspinors of the Dirac operator realizing the case of equality in his lower eigenvalue estimate. An interesting application of imaginary Killing spinors using methods related to Witten's proof of the positive mass theorem has been given in [2]. Recently, a correspondence between twistor and Killing spinors on semi-Riemannian spin manifolds and Killing vector fields in the semi-Riemannian supergeometry canonically associated to a semi-Riemannian spin manifold has been established (see [1]).

Theorem 1 (Killing spinors and curvature, see [7]). *Let $(M^{n,k}, g)$ be a connected semi-Riemannian spin manifold and $\varphi \in \Gamma(S)$ a non-trivial Killing spinor to $\lambda \in \mathbb{C}$. Then φ has no zeroes and the $(1,1)$-Ricci tensor satisfies $\mathrm{Ric}(X) \cdot \varphi = 4\lambda^2(n-1)X \cdot \varphi$ for all vectors $X \in TM$. In particular, the scalar curvature is a constant related to λ by $R = 4n(n-1)\lambda^2$.*

Non-trivial Killing spinors have no zeroes and the space of Killing spinors to λ is of dimension less than or equal to $2^{[n/2]}$. The last formula implies that the Killing number λ of a non-trivial Killing spinor φ has to be real or purely imaginary depending on the scalar curvature R. Accordingly, a Killing spinor $\varphi \in \Gamma(S)$ to $\lambda \in \mathbb{C}$ is called a *real Killing spinor* if $\lambda \in \mathbb{R}\backslash\{0\}$, an *imaginary Killing spinor* if $\lambda \in \mathrm{i}\mathbb{R}\backslash\{0\}$ and a *parallel spinor* if $\lambda = 0$. The model spaces $\mathbb{R}^{n,k}$, $\mathbb{S}^{n,k}$ and $\mathbb{H}^{n,k}$ of constant sectional curvature admit a space of parallel spinors and Killing spinors to $\pm\frac{1}{2}$ or $\pm\frac{1}{2}\mathrm{i}$, respectively, which has the maximal dimension $2^{[n/2]}$.

Furthermore, the theorem implies that a connected Riemannian manifold (M^n, g) admitting a non-trivial solution $\varphi \in \Gamma(S)$ of the Killing equation is an Einstein manifold. This is not true in the case of pseudo-Riemannian manifolds (because the Clifford product of an isotropic vector with a non-zero spinor can be zero). In Section 4, we will see for example that there are non-Einstein Lorentzian spin manifolds admitting parallel spinors and imaginary Killing spinors. However, it will be shown later on that Lorentzian manifolds admitting real Killing spinors are Einstein spaces (see Section 5).

Wang proved in [15] that a non-locally symmetric Riemannian manifold with a parallel spinor has the reduced holonomy of a special type. This has been generalized to the semi-Riemannian case by Baum and Kath (see [6]).

Theorem 2 (Parallel spinors). *Let (M^n, g) be a 1-connected, non-locally symmetric, irreducible semi-Riemannian spin manifold. Denote by $K = \dim \mathcal{K}_0(M, g)$ the dimension of the space of parallel spinors. Then $K > 0$ if and only if the holonomy of M is one of the following*:

1. $\mathrm{SU}(m,k) \subset \mathrm{SO}(2m,2k)$ *and* $K = 2$ (*where* $n = 2m$ *is the dimension and* $2k$ *the index*),
2. $\mathrm{SP}(l,k) \subset \mathrm{SO}(4l,4k)$ *and* $K = l + k + 1$ (*where* $n = 4l$ *is the dimension and* $4k$ *the index*),
3. $\mathrm{G}_2 \subset \mathrm{SO}(7)$ *and* $K = 1$,
4. $\mathrm{G}^*_{2(2)} \subset \mathrm{SO}(7,4)$ *and* $K = 1$,
5. $\mathrm{G}_2^{\mathbb{C}} \subset \mathrm{SO}(14,7)$ *and* $K = 2$,
6. $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ *and* $K = 1$.
7. $\mathrm{Spin}_0(7,4) \subset \mathrm{SO}(8,4)$ *and* $K = 1$.
8. $\mathrm{Spin}(7)^{\mathbb{C}} \subset \mathrm{SO}(16,8)$ *and* $K = 1$.

In the proof of this theorem it is shown that if the holonomy group of a 1-connected semi-Riemannian manifold $(M^{n,k}, g)$—not necessarily being locally symmetric or irreducible—is contained in one of these groups, the dimension of the space of Killing spinors is at least K.

In the Riemannian case it has been shown using warped product techniques that the existence problem for Killing spinors on a Riemannian manifold M^n can be reduced to the existence problem for parallel spinors on a Riemannian manifold either in one dimension lower or in one dimension higher. We now briefly recall the relevant results, because they will be useful in Section 5. The first is the following result obtained by Baum in 1988 (see [7]).

Theorem 3 (Imaginary Killing spinors—Riemannian case). *Let* (M^n, g) *be a complete Riemannian spin manifold with* $R = -n(n-1)$. *If* M *admits a non-trivial imaginary Killing spinor* $\varphi \in \Gamma(S)$ *to* $\lambda = \pm\frac{1}{2}\mathrm{i}$, *then* M *is isometric to a warped product* $F^{n-1}{}_f \times \mathbb{R}$, *where* F *is a complete Riemannian spin manifold admitting a parallel spinor and* $f(t) = \mathrm{e}^{\pm t}$.

In [3], warped product techniques were applied to get a geometrical description of Riemannian manifolds with real Killing spinors. Bär proved that Killing spinors on M correspond to parallel spinors on the cone $\hat{M}$ over M. Using this idea and Wang's result on parallel spinors, he proved the following theorem.

Theorem 4 (Real Killing spinors—Riemannian case). *Let* (M^n, g) *be a* 1*-connected complete Riemannian spin manifold with* $R = n(n-1)$. *Let* $K_\pm = \dim \mathcal{K}_{\pm(1/2)}(M, g)$. *If* K_+ *or* K_- *are non-zero, then one of the following holds*:

1. $M \cong \mathbb{S}^n$ *and* $K_\pm = \dim \Delta_n = 2^{[n/2]}$,
2. $n \equiv 1 \bmod 4$, $n \geq 5$ *and* M *is an Einstein–Sasaki manifold and* $K_\pm = 1$,
3. $n \equiv 3 \bmod 4$, $n \geq 7$ *and*
   - M *is Einstein–Sasaki manifold, but does not carry a Sasaki*-3-*structure and* $K_- = 2$ *and* $K_+ = 0$, *or*
   - M *is a Sasaki*-3-*manifold and* $K_- = \frac{1}{4}(n+5)$ *and* $K_+ = 0$,
4. $n = 6$ *and* M *is a nearly Kähler non-Kähler manifold and* $K_\pm = 1$,
5. $n = 7$ *and* M *admits a nice* 3*-form*(*coming from a* $\mathrm{Spin}(7)$*-holonomy of the cone over* M) *and* $K_- = 1$ *and* $K_+ = 0$.

Conversely, if M is of one of the above types, then M admits Killing spinors and $K_\pm$ *coincide with the given values.*

In this paper, we first describe the general relation between Killing spinors on a semi-Riemannian manifold F and Killing spinors on the warped product $M = F_f \times I$ with an interval I. Using this, we are able to give the following generalization to the semi-Riemannian context of the main idea behind Bär's proof of Theorem 4: real Killing spinors on a semi-Riemannian manifold $M^{n,k}$ correspond to parallel spinors on the cone $\hat{M}^{+1} = M_t \times \mathbb{R}_+^{1,0}$, while imaginary Killing spinors on M correspond to parallel spinors on the cone $\hat{M}^{-1} = M_t \times \mathbb{R}_+^{1,1}$ (with the metric $-\mathrm{d}t^2$ on the interval).

Using the remark following Theorem 2, we can prove that if the holonomy of the cone $\hat{M}^{\pm 1}$ is in the list of Theorem 2, there are Killing spinors on M (provided M is 1-connected). This shows that a 1-connected Lorentzian Einstein–Sasaki manifold admits imaginary Killing spinors. In contrast to the Riemannian case (Theorem 4), the converse is not true, i.e. not all Lorentzian manifolds admitting imaginary Killing spinors are Einstein–Sasaki. We give an example of a non-Einstein Lorentzian manifold admitting an imaginary Killing spinor.

Lorentzian manifolds with real Killing spinors have a local warped product structure (defined almost everywhere) given by the closed conformal vector field associated to the spinor. In contrast to Baum's theorem, the restriction of the real Killing spinor to the fiber of the warped product, which is a codimension one submanifold, can be a real or imaginary Killing spinor or a parallel spinor. (In the case of imaginary Killing spinors on Riemannian manifolds, the restriction is always a parallel spinor.)

2. Killing spinors on warped products

In this section, we deal with the Killing equation on warped products of the form

$$(M^{n,k}, g_M) = F^{n-1,\hat{k}}{}_f \times I = (F \times I,\ f^2(t) g_F + \epsilon\, \mathrm{d}t^2),$$

where $(F^{n-1,\hat{k}}, g_F)$ is an arbitrary semi-Riemannian manifold, $\epsilon \in \{\pm 1\}$, $k = \hat{k} - \frac{1}{2}(\epsilon - 1)$ and $f : I \to \mathbb{R}_+$ is the so-called warping function. We denote by ξ the global unit vector field $\xi = \partial/\partial t$ orthogonal to the fibers of the warped product. For every vector field $X \in \mathfrak{X}(F)$ on F, define $\tilde{X}(p,t) = f^{-1}(t) X(p) \in \mathfrak{X}(M)$.

Lemma 5. *Let* $M^{n,k} = F^{n-1,\hat{k}}{}_f \times I$ *be a semi-Riemannian warped product of dimension* $n \geq 3$. *Then M is an Einstein space (resp. constant curvature space) if and only if the warping function f satisfies the equation* $R^F = \epsilon(n-1)(n-2)((f')^2 - ff')$ *(or the equivalent equation* $R^M = -\epsilon n(n-1) f'' f^{-1}$*) and F is an Einstein space (resp. constant curvature space).*

If $n \geq 4$, *then M is conformally flat if and only if F is of constant sectional curvature.*

The proof is an application of the formulas for the curvature of warped products given in [14, Chapter 7]. Using the corresponding formulas for the spinor calculus on warped products (see 1.20–1.23 of [7]), we prove the main theorems of this section.

Theorem 6. *Let* $M^{2m+2,k} = F^{2m+1,\hat{k}}{}_f \times I$ *be a warped product with metric* $g_M = f^2 g_F + \epsilon\, dt^2$ *and spin structure. Then* M *admits a non-trivial Killing spinor* φ *to the Killing number* $\lambda \in \{\pm\frac{1}{2}, 0, \pm\frac{1}{2}i\}$ *if and only if the warping function satisfies* $f'' = -4\epsilon\lambda^2 f$ *and* F *admits a non-trivial Killing spinor to* $+\lambda_F$ *or* $-\lambda_F$, *where* $\lambda_F^2 = \lambda^2 f^2 + \epsilon\frac{1}{4}(f')^2$. *Furthermore, we have*

$$\dim \mathcal{K}_\lambda(M, g_M) = \dim \mathcal{K}_{+\lambda_F}(F, g_F) + \dim \mathcal{K}_{-\lambda_F}(F, g_F), \qquad \textit{for } \lambda_F \neq 0$$
$$\dim \mathcal{K}_\lambda(M, g_M) \geq \dim \mathcal{K}_0(F, g_F) \quad \textit{for } \lambda_F = 0.$$

Proof. We use the isomorphism $\tilde{}: pr_1^* S_F \oplus pr_1^* \hat{S}_F \to S_M$ described in Chapter 1.2 of [7] (where $\hat{S}_F$ is just another realization of the spinor bundle as in Chapter 1.1 of [7]). Let $\tilde{\varphi} = (\varphi_{1t} + \widehat{\varphi_{2t}})\tilde{} \in \Gamma(S_M)$ be a spinor on a warped product $M = F_f \times I$. It is a Killing spinor to the Killing number λ if and only if $\nabla_\xi \tilde{\varphi} = \lambda \xi \cdot \tilde{\varphi}$ and $\nabla_{\tilde{X}} \tilde{\varphi} = \lambda \tilde{X} \cdot \tilde{\varphi}$ for any $X \in \mathfrak{X}(F)$. Using the formulas $\tilde{X} \cdot \tilde{\varphi} = (X \cdot \varphi_1 - \widehat{X \cdot \varphi_2})\tilde{}$ and $\xi \cdot \tilde{\varphi} = \tau_\epsilon i(-1)^m (\varphi_2 + \hat{\varphi}_1)\tilde{}$ for the Clifford product (where $\tau_\epsilon = \sqrt{\epsilon}$) and $\nabla_{\tilde{X}} \tilde{\varphi} = (1/f)(\nabla_X \varphi_1 + \widehat{\nabla_X \varphi_2})\tilde{} - \epsilon\frac{1}{2}(f'/f)\tilde{X} \cdot \xi \cdot \tilde{\varphi}$ and $\nabla_\xi \tilde{\varphi} = ((\partial/\partial t)\varphi_{1t} + \widehat{(\partial/\partial t)\varphi_{2t}})\tilde{}$ for the spinor derivative one easily sees that this is equivalent to the following equations for t-depending spinors $\varphi_{1t}, \varphi_{2t} \in \Gamma(pr_1^* S_F)$ on F (defining $\kappa = \epsilon\frac{1}{2}\tau_\epsilon i(-1)^m$ and $\omega = \tau_\epsilon i\lambda(-1)^m$ to simplify the formulas)

$$\nabla_X \varphi_{1t} = \lambda f(t) X \cdot \varphi_{1t} + \kappa f'(t) X \cdot \varphi_{2t},$$
$$\nabla_X \varphi_{2t} = -\lambda f(t) X \cdot \varphi_{2t} - \kappa f'(t) X \cdot \varphi_{1t} \quad \text{for } t \in I, X \in \mathfrak{X}(F), \tag{I}$$

$$\frac{\partial}{\partial t}\varphi_{1t} = \omega\varphi_{2t}, \qquad \frac{\partial}{\partial t}\varphi_{2t} = \omega\varphi_{1t} \quad \text{for any } t \in \mathbb{R}. \tag{II}$$

If we have a solution $\varphi_{1t_0}, \varphi_{2t_0}$ of Eq. (I) for $t_0 \in I$, i.e.

$$\nabla_X \varphi_{1t_0} = \lambda f(t_0) X \cdot \varphi_{1t_0} + \kappa f'(t_0) X \cdot \varphi_{2t_0},$$
$$\nabla_X \varphi_{2t_0} = -\lambda f(t_0) X \cdot \varphi_{2t_0} - \kappa f'(t_0) X \cdot \varphi_{1t_0} \quad \text{for all } X \in \mathfrak{X}(F), \tag{I'}$$

then a solution of Eq. (II) is given by

$$\varphi_{1t} = \cosh(\omega(t - t_0))\varphi_{1t_0} + \sinh(\omega(t - t_0))\varphi_{2t_0},$$
$$\varphi_{2t} = \sinh(\omega(t - t_0))\varphi_{1t_0} + \cosh(\omega(t - t_0))\varphi_{2t_0}. \tag{*}$$

The proof of the theorem is cut into three parts. First we show that given a Killing spinor on M, the warping function necessarily satisfies $f'' = -4\epsilon\lambda^2 f$. Then we prove that if f satisfies this differential equation, any solution $\varphi_{1t_0}, \varphi_{2t_0}$ of (I′) by (*) defines a solution of (I) for all $t \in I$. The third part is to analyze the relation between Eq. (I′) and the Killing equation.

First part: By (I) and Theorem 2 of [7] it is clear that φ_{1t} and φ_{2t} are twistor spinors on F. Thus, by 1.33 of [7], we have

$$\mathcal{D}^2\varphi_{1t} = \frac{1}{4}R^F\frac{n-1}{n-2}\varphi_{1t}.$$

On the other hand, a direct computation using (I) shows

$$\mathcal{D}^2\varphi_{1t} = (n-1)^2(\lambda^2 f^2(t) - \kappa^2(f')^2(t))\varphi_{1t}.$$

Supposing φ_{1t} to be non-trivial (otherwise proceeding similarly with φ_{2t}), we get

$$R^F = 4(n-1)(n-2)(\lambda^2 f^2(t) - \kappa^2(f')^2(t)). \tag{1}$$

The equation $R^M = (1/f^2)R^F - \epsilon(n-1)((n-2)(f'/f)^2 + 2(f''/f))$ together with $R^M = 4n(n-1)\lambda^2$ (see Theorem 1) yields $f'' = -4\epsilon\lambda^2 f$.

Second part: If the warping function satisfies $f'' = -4\epsilon\lambda^2 f$, one can extend a given solution of Eq. (I′) for t_0 by (*) to a solution of (I) for all $t \in I$. We will only prove that the first part of (I) is satisfied. Therefore, we have to prove that the expression

$$\begin{aligned}\nabla_X\varphi_{1t} &= \cosh(\omega(t-t_0))\{\lambda f(t_0)X\cdot\varphi_{1t_0} + \kappa f'(t_0)X\cdot\varphi_{2t_0}\}\\ &\quad+\sinh(\omega(t-t_0))\{-\lambda f(t_0)X\cdot\varphi_{2t_0} - \kappa f'(t_0)X\cdot\varphi_{1t_0}\}\end{aligned}$$

is equal to

$$\begin{aligned}&\lambda f(t)X\cdot\{\cosh(\omega(t-t_0))\varphi_{1t_0} + \sinh(\omega(t-t_0))\varphi_{2t_0}\}\\ &\quad+\kappa f'(t)X\cdot\{\sinh(\omega(t-t_0))\varphi_{1t_0} + \cosh(\omega(t-t_0))\varphi_{2t_0}\}.\end{aligned}$$

This can be done by comparing the coefficients of $X\cdot\varphi_{1t_0}$ and $X\cdot\varphi_{2t_0}$ separately. Since all coefficients satisfy the second-order differential equation $x'' = -\epsilon\lambda^2 x$ and corresponding coefficients coincide for t_0 as well as their first derivatives with respect to t, both expressions are equal for all $t \in I$.

Thus, via (*) a solution of Eq. (I′) for t_0 defines a solution of (I) for all $t \in I$ and therefore $\tilde{\varphi} = (\varphi_{1t} + \widehat{\varphi_{2t}})\tilde{} \in \Gamma(S_M)$ solves the Killing equation on M.

Third part: We now give an interpretation of Eq. (I′) in terms of solutions of the Killing equation. In the first part, we derived $R^F = 4(n-1)(n-2)(\lambda^2 f^2(t) - \kappa^2(f')^2(t))$. Thus, the only possible Killing numbers on F are $\pm\lambda_F$, where

$$\lambda_F^2 = \frac{R^F}{4(n-1)(n-2)} = \lambda^2 f(t_0)^2 + \epsilon\frac{1}{4}(f')^2(t_0).$$

In the case $f'(t_0) = 0$ nothing is to be done (because then a solution of (I′) corresponds to a pair of Killing spinors to $\pm\lambda$ and we have $\lambda_F \in \{\pm\lambda\}$). So in the following we suppose $f'(t_0) \neq 0$.

Case ($R^F \neq 0$ (or equivalently $\lambda_F \neq 0$)). Given a solution $\varphi_{1t_0}, \varphi_{2t_0}$ of Eq. (I′)

$$\psi_+ = \varphi_{1t_0} - \frac{\lambda_F - \lambda f(t_0)}{\kappa f'(t_0)}\varphi_{2t_0}, \qquad \psi_- = \varphi_{1t_0} - \frac{-\lambda_F - \lambda f(t_0)}{\kappa f'(t_0)}\varphi_{2t_0}$$

defines a pair of Killing spinors $\psi_\pm$ to $\pm\lambda_F$. Conversely, starting with a pair of Killing spinors $\psi_\pm$ to $\pm\lambda_F$, one gets a solution of Eq. (I′) by setting

$$\varphi_{1t_0} = \frac{\lambda_F + \lambda f(t_0)}{2\lambda_F}\psi_+ + \frac{\lambda_F - \lambda f(t_0)}{2\lambda_F}\psi_-, \qquad \varphi_{2t_0} = -\frac{\kappa f'(t_0)}{2\lambda_F}\psi_+ + \frac{\kappa f'(t_0)}{2\lambda_F}\psi_-.$$

Both maps are inverse to each other.

Case ($R^F = 0$ (or equivalently $\lambda_F = 0$)). Given a solution $\varphi_{1t_0}, \varphi_{2t_0}$ of Eq. (I′),

$$\psi = \varphi_{1t_0} + \frac{\lambda f(t_0)}{\kappa f'(t_0)}\varphi_{2t_0}$$

defines a parallel spinor ψ on F. If it is trivial, $\psi = \varphi_{1t_0}$ is already a parallel spinor. Conversely, a parallel spinor ψ on F defines a solution of Eq. (I′) by

$$\varphi_{1t_0} = \psi, \qquad \varphi_{2t_0} = -\frac{\lambda f(t_0)}{\kappa f'(t_0)}\psi.$$

Theorem 7. *Let $M^{2m+1,k} = F^{2m,\hat{k}}{}_f \times I$ be a warped product with the metric $g_M = f^2 g_F + \epsilon\, dt^2$ and spin structure. Then M admits a non-trivial Killing spinor φ to the Killing number $\lambda \in \{\pm\frac{1}{2}, 0, \pm\frac{1}{2}i\}$ if and only if the warping function satisfies $f'' = -4\epsilon\lambda^2 f$ and F admits a non-trivial Killing spinor to $\pm\lambda_F$, where $\lambda_F^2 = \lambda^2 f^2 + \epsilon\frac{1}{4}(f')^2$. Furthermore, we have*

$$\dim \mathcal{K}_\lambda(M, g_M) = \dim \mathcal{K}_{\pm\lambda_F}(F, g_F) \quad \textit{if } \lambda_F \neq 0.$$

In the case $\lambda_F = 0$, we have $\lambda = \pm\kappa(f'(t_0)/f(t_0))$ with $\kappa = \epsilon\frac{1}{2}\tau_\epsilon i(-1)^m$ and

$$\dim \mathcal{K}_{+\kappa(f'(t_0)/f(t_0))}(M, g_M) \geq \dim \mathcal{K}_0^-(F, g_F),$$
$$\dim \mathcal{K}_{-\kappa(f'(t_0)/f(t_0))}(M, g_M) \geq \dim \mathcal{K}_0^+(F, g_F),$$

where $\mathcal{K}_0^\pm(F, g_F) = \{\varphi \in \Gamma(S_F^\pm) : \nabla\varphi = 0\}$.

Proof. The proof is analogous to that of the preceding theorem. □

An application: the cone over a manifold. Here, we slightly change the point of view and consider the fiber F as the object of main interest. We therefore use another notation than before. Let $(M^{n,k}, g)$ be a semi-Riemannian manifold and let $\epsilon \in \{\pm 1\}$. The warped product $\hat{M}^\epsilon = M_t \times \mathbb{R}^+$ with metric $g_{\hat{M}} = t^2 g_M + \epsilon\, dt^2$ is called the ϵ-*cone* over the semi-Riemannian manifold M. The following corollary is an easy consequence of the preceding theorems.

Corollary 8. *Let $(M^{n,k}, g_M)$ be a semi-Riemannian spin manifold and let $\tau_\epsilon = \sqrt{\epsilon}$. Then*

$$\dim \mathcal{K}_{\pm(\tau_\epsilon/2)}(M, g_M) = \dim \mathcal{K}_0(\hat{M}^\epsilon, g_{\hat{M}}) \quad \textit{if } n \textit{ is even},$$
$$\dim \mathcal{K}_{+(\tau_\epsilon/2)}(M, g_M) + \dim \mathcal{K}_{-(\tau_\epsilon/2)}(M, g_M) = \dim \mathcal{K}_0(\hat{M}^\epsilon, g_{\hat{M}}) \quad \textit{if } n \textit{ is odd}.$$

Thus, real and imaginary Killing spinors correspond to parallel spinors on the cones over the manifold. This is a generalization of the basic idea behind the proof of Theorem 4. Since the 1-cone over $\mathbb{S}^{n,k}$ and the -1-cone over $\mathbb{H}^{n,k}$ are isometric to open subsets of $\mathbb{R}^{n+1,k}$, resp., $\mathbb{R}^{n+1,k+1}$, the existence of real, resp., imaginary Killing spinors on the non-flat model spaces follows from the existence of parallel spinors on $\mathbb{R}^{n+1,k}$, resp., $\mathbb{R}^{n+1,k+1}$.

3. Imaginary Killing spinors on Lorentzian Einstein–Sasaki manifolds

Friedrich and Kath found that 1-connected Einstein–Sasaki manifolds admit at least two linearly independent Killing spinors (see [7]). Here, we generalize this result to the Lorentzian case. Another approach for doing this can be found in [11]. In the Riemannian case, Theorem 4 asserts that — except in dimension 7 — all odd-dimensional complete 1-connected manifolds admitting real Killing spinors are Einstein–Sasaki (up to rescaling of the metric). This is not true for imaginary Killing spinors on Lorentzian manifolds.

3.1. Pseudo-Sasaki manifolds

Definition 9. A semi-Riemannian manifold $(M^{2m+1,k}, g)$ is called *pseudo-Sasaki manifold* if there is a vector field $\xi \in \mathfrak{X}(M)$ which defines a *pseudo-Sasaki structure*, i.e. which satisfies the following:

1. ξ is a Killing vector field of length $\epsilon = g(\xi, \xi) = (-1)^k$,
2. the endomorphism $\phi = -\nabla\xi$ satisfies $\phi^2(X) = -X + \epsilon g(X, \xi)\xi$ for all vectors $X \in TM$, and
3. $(\nabla_X \phi)(Y) = \epsilon g(X, Y)\xi - \epsilon g(Y, \xi)X$ for any $X, Y \in TM$.

A Riemannian manifold is Sasaki if and only if the Riemannian cone over the manifold is Kähler (see [3]). This can be generalized as follows.

Lemma 10 (Cone over pseudo-Sasaki manifold). *Let $(M^{2m+1,k}, g_M)$ be a semi-Riemannian manifold and $\epsilon = (-1)^k$. Denote by $\hat{M}^\epsilon = M_t \times \mathbb{R}^+$ the ϵ-cone over M (with the metric $g_{\hat{M}} = t^2 g_M + \epsilon\, dt^2$). Then there is a one–one-correspondence between:*

1. *Pseudo-Sasaki structures $\xi \in \mathfrak{X}(M)$ with $\epsilon = g(\xi, \xi) = (-1)^k$ on (M, g_M).*
2. *Pseudo-Kähler structure $J \in \Gamma(\mathrm{End}\, T\hat{M})$ on $\hat{M}$, i.e. endomorphisms satisfying*

$$J^2 = -\mathrm{Id}, \qquad J^* g_{\hat{M}} = g_{\hat{M}}, \qquad \nabla J = 0.$$

Proof. Given ξ, the almost complex structure J is defined by $J\tilde{X} = -\widetilde{\phi(X)}$ for $X \in \xi^\perp$, $J\tilde{\xi} = -(\partial/\partial t)$ and $J(\partial/\partial t) = \tilde{\xi}$. Conversely, given J, define $\xi = J(\partial/\partial t)$ on $M = M \times \{1\} \subset \tilde{M}$. □

Lemma 5 implies that M is Einstein if and only if $\hat{M}$ is Ricci-flat and that M is of constant sectional curvature if and only if $\hat{M}$ is flat. A simple computation shows that if M is Einstein or of constant sectional curvature, then $R = \epsilon(2m + 1)2m$.

Theorem 11 (Killing spinors on Einstein pseudo-Sasaki manifolds). *Let $(M^{2m+1,k}, g, \xi)$ be a 1-connected Einstein pseudo-Sasaki spin manifold (with $m \geq 2$), $\epsilon = g(\xi, \xi) = (-1)^k$.*

Denote by $K_\pm = \dim \mathcal{K}_{\pm(\tau_\epsilon/2)}(M, g)$. *Then, the minimal values for* $K_\pm$ *are given by the following table:*

	$\epsilon = 1$	$\epsilon = -1$
m odd		
K_+	0	2
K_-	2	0
m even		
K_+	1	1
K_-	1	1

Proof. We have seen that a pseudo-Sasaki structure corresponds to a unique pseudo-Kähler structure on the cone over the manifold. In the case of Einstein pseudo-Sasaki manifolds, this pseudo-Kähler structure is Ricci-flat. A Ricci-flat pseudo-Kähler structure on a 1-connected manifold corresponds to a unique reduction of the Repère bundle and the Levi-Civita connection to $\mathrm{SU}(m, k)$ (see [12, Theorem 4.6 of Chapter 10] or [8, Theorem 10.29]). Then the holonomy of the cone is a subgroup of $\mathrm{SU}(m, k)$ and we can apply the remark following Theorem 2 and Corollary 8. □

3.2. Lorentzian Einstein–Sasaki manifolds

The next lemma shows that any 1-connected Lorentzian Einstein–Sasaki manifold is spin. The preceding theorem therefore implies that any such manifold admits imaginary Killing spinors.

Lemma 12. *Any* 1*-connected Lorentzian Einstein–Sasaki manifold* $(M^{2m+1,1}, g, \xi)$ *is spin.*

Proof. We have seen that a Lorentzian–Sasaki structure on M corresponds to a pseudo-Kähler structure on the cone $\hat{M}^{-1}$. Since M is Einstein, $\hat{M}^{-1}$ is Ricci-flat. Now, the same argument as used in the proof of Theorem 11 implies that the holonomy of $\hat{M}^{-1}$ is contained in $\mathrm{SU}(m+1, 1)$.

Since there are the two vector fields $\partial/\partial t$ and $\tilde{\xi} = J(\partial/\partial t)$ on $\hat{M}^{-1}$, we not only have a reduction to $\mathrm{SU}(m+1, 1)$, but also to $\mathrm{SU}(m)$. Now in the commutative diagram of group monomorphisms

$$\begin{array}{ccc} \mathrm{SU}(m) & \rightarrow & \mathrm{SO}(2m) \\ \downarrow & & \downarrow \\ \mathrm{SU}(m+1,1) & \rightarrow & \mathrm{SO}_0(2m+2,2), \end{array}$$

the upper arrow factor over $\mathrm{Spin}(2m)$ (see [10, Chapter 1.6]). Therefore, the cone $\hat{M}^{-1}$ admits a $\mathrm{Spin}(2m)$- and by extension a $\mathrm{Spin}_0(2m+2, 2k)$-structure. But a Spin structure on the cone $\hat{M}^{-1}$ defines a spin structure on M. □

As we already pointed out, not all Lorentzian manifolds admitting imaginary Killing spinors are Einstein–Sasaki. The following theorem gives a spinorial characterization of 1-connected Lorentzian Einstein–Sasaki manifolds. A proof can be found in [9].

Theorem 13. *Let $(M^{2m+1,1}, g, \xi)$ be a 1-connected Lorentzian Einstein–Sasaki spin manifold. For any spinor $\varphi \in \Gamma(S)$, denote by V_φ the associated vector field of φ, i.e. the vector field dual to the 1-form $\omega_\varphi(X) = -\langle \varphi, X \cdot \varphi \rangle$. If m is even, then there is a Killing spinor $\varphi \in \Gamma(S)$ to each $\lambda \in \{\pm \frac{1}{2}\mathrm{i}\}$ such that $V_\varphi = \xi$ and satisfies $V_\varphi \cdot \varphi = -\varphi$. If m is odd, then there are two Killing spinors $\varphi_1, \varphi_2 \in \Gamma(S)$ to $\frac{1}{2}\mathrm{i}$ such that $V_{\varphi_1} = V_{\varphi_2} = \xi$ and $\xi \cdot \varphi_1 = \varphi_1$ and $\xi \cdot \varphi_2 = -\varphi_2$.*

Conversely, if a Lorentzian spin manifold $(M^{2m+1,1}, g)$ admits a Killing spinor $\varphi \in \Gamma(S)$ to $\lambda \in \{\pm \frac{1}{2}\mathrm{i}\}$ such that $g(V_\varphi, V_\varphi) = -1$ and $V_\varphi \cdot \varphi = \pm\varphi$, then M is Einstein and $\xi = V_\varphi$ defines a Lorentzian–Sasaki structure.

3.3. *Examples of Lorentzian Einstein–Sasaki manifolds*

Examples of Lorentzian Einstein–Sasaki manifolds can be obtained using a method for the construction of Riemannian Einstein–Sasaki manifolds admitting real Killing spinors (see [7, Chapter 4.2, Example 1]). Starting with a Kähler–Einstein manifold X^{2m} of positive scalar curvature, a Riemannian Einstein–Sasaki manifold can be obtained by taking a suitable metric on a special $\mathbb{S}^1$-bundle over X^{2m}. In contrast to the Riemannian case one has to start with Kähler–Einstein manifolds of negative scalar curvature in order to get Lorentzian–Sasaki manifolds admitting imaginary Killing spinors.

Lemma 14 ($\mathbb{S}^1$-bundle and its metric). *Let (X^{2m}, g_X, J) be a Kähler–Einstein manifold with $R_X = -4m(m+1)$. Let $M^{2m+1} \xrightarrow{\pi} X^{2m}$ be an $\mathbb{S}^1$-bundle over X^{2m} with first Chern class*

$$c_1(M^{2m+1} \xrightarrow{\pi} X^{2m}) = \frac{1}{A} c_1(X^{2m}) \in H^2(X^{2m}, \mathbb{Z}),$$

where $A \in \mathbb{N}$ is maximal such that $(1/A)c_1(X^{2m})$ is integral. Taking a connection η' whose curvature is related to the Kähler form $\Omega = g_X(\cdot, J\cdot)$ of X^{2m} by $\mathrm{d}\eta' = -(2(m+1)/A)\mathrm{i}\Omega$, an Einstein metric on M can be defined by

$$g_M = \pi^* g_X + \epsilon \frac{A^2}{(m+1)^2} \eta' \otimes \eta'.$$

Furthermore, there is a canonical Lorentzian–Sasaki structure defined by the Killing vector field

$$\xi = \left(\frac{m+1}{A} \mathrm{i} \right)^{\sim}$$

of length $g_M(\xi, \xi) = \epsilon$. (Here $\tilde{}$ denotes the fundamental vector field with respect to the $\mathbb{S}^1$-action of the principal fiber bundle.)

The proof of this lemma is analogous to Example 1 in Chapter 4.2 of [7].

Now that we have a construction principle for Lorentzian Einstein–Sasaki manifolds as $\mathbb{S}^1$-bundles over Kähler–Einstein manifolds of negative scalar curvature, we want to apply Theorem 11. In order to do that, we only have to assure that the Einstein–Sasaki manifold

M is 1-connected, i.e. if the constructed manifold is not 1-connected, we have to consider its universal covering. By Lemma 12 and Theorem 11, this 1-connected manifold admits imaginary Killing spinors.

In order to construct compact 1-connected Lorentzian Einstein–Sasaki manifolds, we need the following lemma.

Lemma 15. *Let (X^{2m}, J, g_X) be a compact and 1-connected Kähler–Einstein manifold of scalar curvature $R_X = -4m(m+1)$. Let $M^{2m+1} \xrightarrow{\pi} X^{2m}$ be the Lorentzian Einstein–Sasaki manifold obtained as $\mathbb{S}^1$-bundle over X^{2m} as in Lemma 14. Then M is a compact 1-connected spin manifold.*

Proof. M is compact, because it is a principal fiber bundle with compact structure group $\mathbb{S}^1$ over a compact base X. To prove that M again is 1-connected, one can proceed as in [7, Chapter 4.2, Example 1]. Now M is spin by Lemma 12. □

The Aubin–Calabi–Yau theorem (cf. [8, 11.17]) asserts that any compact complex manifold (X^{2m}, J) of negative first Chern class admits a unique (up to scaling) Kähler–Einstein metric of negative scalar curvature. One class of 1-connected compact complex manifolds of negative first Chern class is given in [8, 11.10]. Using this class and Lemmas 14 and 15, one can construct 1-connected compact Lorentzian Einstein–Sasaki manifolds with imaginary Killing spinors.

4. A non-Einstein Lorentzian manifold with imaginary Killing spinors

Here, we give an example of a non-Einstein Lorentzian manifold admitting imaginary Killing spinors. This is interesting because in the Riemannian case all manifolds admitting Killing spinors are Einstein. In the Lorentzian case, only the manifolds admitting real Killing spinors have to be Einstein (see Section 5).

Baum proved in [5] that in dimension $n \geq 3$ the 1-connected undecomposable solvable Lorentzian symmetric spaces, which are all of the form $(\mathbb{R}^n, g_{\underline{\lambda}})$, where $g_{\underline{\lambda}} = 2\,\mathrm{d}s\,\mathrm{d}t + \sum_{j=1}^{n-2}\lambda_j x_j^2\,\mathrm{d}s^2 + \sum_{j=1}^{n-2}\mathrm{d}x_j^2$ and $\underline{\lambda} = (\lambda_1, \dots, \lambda_{n-2}) \in (\mathbb{R}\backslash\{0\})^{n-2}$, admit a space of parallel spinors of dimension $\frac{1}{2}\cdot 2^{[n/2]}$. The vector field $V = \partial/\partial t$ is isotropic and parallel. The Ricci tensor satisfies $\mathrm{Ric}(X) = (-\sum_{j=1}^{n-2}\lambda_j)g_{\underline{\lambda}}(X, V)V$. Such spaces are Einstein if and only if $\sum_{j=1}^{n-2}\lambda_j = 0$ and conformally flat if and only if $\underline{\lambda} = (\lambda, \dots, \lambda)$ for $\lambda \in \mathbb{R}\backslash\{0\}$.

Let $(F, g_F) = (\mathbb{R}^{n-1}, g_{\underline{\lambda}})$ be such a Lorentzian symmetric space with $\underline{\lambda} \in (\mathbb{R}\backslash\{0\})^{n-3}$ such that $\sum_j^{n-3}\lambda_j \neq 0$. Then F is non-Einstein and admits parallel spinors. Lemma 5 implies that $M = F_e \times \mathbb{R}$ is non-Einstein non-conformally flat. But Theorems 6 and 7 show that M admits imaginary Killing spinors. Thus, we have proved the following theorem.

Theorem 16. *In dimensions $n \geq 4$, there are non-Einstein non-conformally flat Lorentzian spin manifolds $(M^{n,1}, g)$ admitting imaginary Killing spinors.*

5. Real Killing spinors on Lorentzian manifolds

This section is devoted to real Killing spinors on Lorentzian manifolds. Given a Killing spinor φ, its associated vector field V_φ is closed and conformal and it is orthogonal to the foliation into the level surfaces of the length function $u_\varphi = \langle\varphi, \varphi\rangle$. This gives rise to a local warped product structure defined outside the hypersurfaces where the vector field is isotropic. We examine more closely, how far this warped product structure can be extended in the case of complete manifolds. At the end, we give a list of all complete 1-connected Lorentzian manifolds admitting a real Killing spinor with first integral $Q_\varphi < 0$. This is done using the classification results for complete 1-connected Riemannian manifolds with real Killing spinors.

5.1. Local warped product structure

Let $(M^{n,1}, g)$ be a connected Lorentzian spin manifold. Let $\varphi \in \Gamma(S)$ be a real Killing spinor to Killing number $\lambda \in \mathbb{R}\backslash\{0\}$. The associated vector field V_φ of the spinor $\varphi \in \Gamma(S)$ is defined to be the vector field dual to the 1-form $\omega_\varphi(X) = -\langle\varphi, X \cdot \varphi\rangle$. In [4], it is proved that on a Lorentzian spin manifold $M^{n,1}$, the vector field V_φ has the same zeroes as the spinor $\varphi \in \Gamma(S)$. So, the associated vector field V_φ of a non-trivial Killing spinor φ has no zeroes at all.

Lemma 17. *The associated vector field V_φ of a real Killing spinor $\varphi \in \Gamma(S)$ to the Killing number $\lambda \in \mathbb{R}\backslash\{0\}$ on a Lorentzian spin manifold $(M^{n,1}, g)$ satisfies*

$$\nabla_X V_\varphi = 2\lambda u X,$$

where $u = \langle\varphi, \varphi\rangle$ is the length function of the Killing spinor. Thus V_φ is a closed conformal vector field. Furthermore, for the length function u, we have

$$\operatorname{grad}(u) = -2\lambda V_\varphi.$$

Proof. Let $s_1, \dots, s_n$ be a local orthonormal frame, which is parallel in $p \in M$. In p, we have

$$\begin{aligned}\nabla_X V_\varphi &= -\sum_i \epsilon_i (\langle s_i \cdot \nabla_X\varphi, \varphi\rangle + \langle s_i \cdot \varphi, \nabla_X\varphi\rangle) s_i = -\lambda \sum_i \epsilon_i \langle (s_i \cdot X + X \cdot s_i) \cdot \varphi, \varphi\rangle s_i \\ &= 2\lambda \sum_i \epsilon_i \langle\varphi, \varphi\rangle g(X, s_i) s_i = 2\lambda u X.\end{aligned}$$

The second formula follows from $X(u) = \langle\nabla_X\varphi, \varphi\rangle + \langle\varphi, \nabla_X\varphi\rangle = -2\lambda\omega_\varphi(X)$. □

The length function u of a non-trivial real Killing spinor φ is regular, because its gradient is up to a constant equal to the vector field V_φ, which has no zeros. Thus, the real Killing spinor φ defines a foliation of the manifold $M^{n,1}$. We now derive a local description of M arising from that foliation. It follows basically from the existence of the conformal gradient vector field. Similar techniques are used in [13].

Corollary 18. *Let $\varphi \in \Gamma(S)$ be a real Killing spinor on a connected Lorentzian spin manifold, denote by $u = \langle\varphi, \varphi\rangle$ its length function and by $v = g(V_\varphi, V_\varphi)$ the length function of its associated vector field. Then $Q_\varphi = u^2 + v$ is constant on M.*

Proof. The preceding lemma yields $X(Q_\varphi) = 2ug(\mathrm{grad}(u), X) + 2g(\nabla_X V_\varphi, V_\varphi) = 0$. □

Thus, the length v of V_φ is constant along the level surfaces of u. Furthermore, the gradient of u — being up to a constant equal to the conformal vector field V_φ — gets only isotropic on the level surfaces $u = \pm\sqrt{Q_\varphi}$, i.e. $v^{-1}(0) = u^{-1}(\pm\sqrt{Q_\varphi})$.

By rescaling, we can assume that the Killing number λ is equal to $\pm\frac{1}{2}$. This simplifies Lemma 17 to

$$\nabla_X V_\varphi = \pm uX, \qquad \mathrm{grad}(u) = \mp V_\varphi. \tag{2}$$

The normalized gradient field ξ is given by

$$\xi = \frac{\mathrm{grad}(u)}{\sqrt{\epsilon g(\mathrm{grad}(u), \mathrm{grad}(u))}} = \mp\frac{V_\varphi}{\sqrt{\epsilon v}}$$

is defined on the open sets $\mathcal{V}_\pm = \{p \in M : v(p) \gtrless 0\} = \{p \in M : Q_\varphi \gtrless u^2(p)\}$. In the following, define $\epsilon : \mathcal{V}_+ \cup \mathcal{V}_- \to \{\pm 1\},\ p \mapsto \mathrm{sgn}(v(p))$ to be the locally constant sign of v. The normalized gradient field ξ is lightlike on $\mathcal{V}_-$ and spacelike on $\mathcal{V}_+$. The union of $\mathcal{V}_+$ and $\mathcal{V}_-$ is dense, because its complement consists of hypersurfaces.

By Corollary 18, there are three possibilities depending on the first integral Q_φ:

- $Q_\varphi < 0$ and $M = \mathcal{V}_-$,
- $Q_\varphi = 0$ and $M = \mathcal{V}_- \dot\cup v^{-1}(0)$ or
- $Q_\varphi > 0$ and $M = \mathcal{V}_- \dot\cup v^{-1}(0) \dot\cup \mathcal{V}_+$.

Using Eq. (2), one can see that ξ satisfies

$$\xi(u) = \epsilon\sqrt{\epsilon v}, \qquad \nabla_X \xi = -\frac{u}{\sqrt{\epsilon v}}\,\mathrm{proj}_{\xi^\perp}(X). \tag{3}$$

The last equation implies that ξ is closed (i.e. $\mathrm{rot}(V) = g(\nabla \cdot \xi, \cdot) - g(\cdot, \nabla \cdot \xi) = 0$) and geodesic (i.e. $\nabla_\xi \xi = 0$). Consequently, the following lemma applies to the integral curves of ξ.

Lemma 19. *Let $(M^{n,1}, g)$ be a Lorentzian spin manifold and let $u = \langle\varphi, \varphi\rangle$ be the length function of a real Killing spinor $\varphi \in \Gamma(S)$ to $\pm\frac{1}{2}$. Let γ be an arbitrary geodesic having the length $\epsilon = g(\gamma'(0), \gamma'(0)) \in \{-1, 0, 1\}$. For the length function $u(t) = u(\gamma(t))$ of the spinor φ along γ, the following holds:*

$$u(t) = \begin{cases} u(0)\cos(t) + u'(0)\sin(t), & \epsilon = 1, \\ u(0) + u'(0)t, & \epsilon = 0, \\ u(0)\cosh(t) + u'(0)\sinh(t), & \epsilon = -1. \end{cases}$$

Proof. Eq. (2) implies that u along γ has to satisfy the differential equation

$$\begin{aligned} u''(t) &= \gamma'(t)g(\mathrm{grad}(u), \gamma'(t)) = g(\nabla_{\gamma'(t)}\,\mathrm{grad}(u), \gamma'(t)) \\ &= -g(\gamma'(t), \gamma'(t))u(t) = -\epsilon u(t). \end{aligned}$$ □

Lemma 20. *Let ξ be the normalized gradient field of the length function u of a real Killing spinor φ to $\pm\frac{1}{2}$. Let Φ_t be the local flow of ξ. Then we have $\mathrm{d}\Phi_t(\xi) = \xi$ and $\mathrm{d}\Phi_t(\xi^{\perp}) = \xi^{\perp}$ for any t where these expressions are defined. Furthermore, for any $X, Y \in \xi_p^{\perp}$, we have*

$$\Phi_t^* g(X, Y) = \mathrm{e}^{-2\int_0^t (u/\sqrt{\epsilon v})(\Phi_s(p))\,\mathrm{d}s} \cdot g_p(X, Y) = \left(\frac{u'(t)}{u'(0)}\right)^2 g_p(X, Y).$$

Proof. As Φ_t is the local flow of ξ, we have $\mathrm{d}\Phi_t(\xi) = \xi$. So let $X, Y \in T_pM$ be arbitrary tangent vectors. Define $a(t) = \Phi_t^* g(X, Y)$. By Eq. (3), the first derivative of a satisfies

$$\begin{aligned} a'(t) &= \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}(\Phi_s^* g)_{\Phi_t(p)}(\mathrm{d}\Phi_t(X), \mathrm{d}\Phi_t(Y)) = (\pounds_\xi g)_{\Phi_t(p)}(\mathrm{d}\Phi_t(X), \mathrm{d}\Phi_t(Y)) \\ &= -\frac{2u}{\sqrt{\epsilon v}}(\Phi_t(p)) g_{\Phi_t(p)}(\mathrm{proj}_{\xi^\perp}(\mathrm{d}\Phi_t(X)), \mathrm{proj}_{\xi^\perp}(\mathrm{d}\Phi_t(Y))). \end{aligned}$$

In the case $X = \xi$ or $Y = \xi$, we have $a'(t) = 0$ and $\mathrm{d}\Phi_t(\xi^\perp) = \xi^\perp$ holds. In the case $X, Y \in \xi^\perp$, we therefore get $\mathrm{proj}_{\xi^\perp}(\mathrm{d}\Phi_t(X)) = \mathrm{d}\Phi_t(X)$. So, $a(t)$ satisfies the linear differential equation

$$a'(t) = -\frac{2u}{\sqrt{\epsilon v}}(\Phi_t(p))a(t)$$

being solved by $\Phi_t^* g(X, Y) = \mathrm{e}^{-2\int_0^t (u/\sqrt{\epsilon v})(\Phi_s(p))\,\mathrm{d}s} \cdot g_p(X, Y)$. Using Eq. (3) and Lemma 19, we have $u' = \epsilon\sqrt{\epsilon v}$ and $u'' = -\epsilon u$. So, we get

$$\Phi_t^* g(X, Y) = \mathrm{e}^{2\int_0^t (u''/u')(\Phi_s(p))\,\mathrm{d}s} \cdot g_p(X, Y) = \left(\frac{u'(t)}{u'(0)}\right)^2 g_p(X, Y).$$ □

By definition, the level surfaces of u are integral manifolds of the geometric distribution given by $\xi^\perp$. The preceding lemma implies that Φ_t, whenever defined on a connected piece of level surface, maps that piece of level surface on another connected piece of level surface. Thus, on the dense set $\mathcal{V}_+ \cup \mathcal{V}_-$, we have the following local form of M.

Lemma 21. *Let $(M^{n,1}, g)$ be a connected Lorentzian spin manifold admitting a non-trivial real Killing spinor φ and let $u = \langle\varphi, \varphi\rangle$. Let $p \in M^{n,1}$ such that $v(p) \neq 0$. Then there is a connected open neighborhood $\mathcal{V}_p \subseteq \mathcal{V}_\epsilon$ ($\epsilon = \mathrm{sgn}(v(p))$) isometric to the warped product $(F^{n-1} \times (-\delta, \delta), f^2(t)g_{|F} + \epsilon\,\mathrm{d}t^2)$, where $F^{n-1} = \mathcal{V} \cap u^{-1}(u(p))$ with the warping function $f(t) = u'(t)/u'(0)$, where $u(t) = u(\Phi_t(p))$.*

Proof. From the theory of differential equations we know that there is an open connected set $F \subseteq u^{-1}(u(p))$ and a $\delta > 0$ such that $\Psi : F \times (-\delta, \delta) \to M, (p', t) \mapsto \Phi_t(p')$ is a diffeomorphism on its image. Define $\mathcal{V}$ to be the image of Ψ. Now $u(t) = u(\Phi_t(p'))$ is independent on the choice of $p' \in F$, because Φ_t maps on connected components of level surfaces. From the preceding lemma, we know that the pullback of the metric on M via Ψ has the form $f^2(t)g_{|F} + \epsilon\, \mathrm{d}t^2$ with the warping function $f(t) = u'(t)/u'(0)$, where $u(t) = u(\Phi_t(p))$. □

This local result can be globalized by the following observation. Every time we can choose F and δ in the way that the map Ψ as in the preceding proof is defined, Ψ is automatically a diffeomorphism on its image, because it is an injective local diffeomorphism (Φ_t is a diffeomorphism and u is strictly monotonous along integral curves of ξ by Eq. (3)).

Theorem 22 (Local warped product structure). *Let $M^{n,1}$ be a Lorentzian spin manifold admitting a non-trivial real Killing spinor φ to $\pm\frac{1}{2}$, let $p \in \mathcal{V}_\pm$ and $\epsilon = \mathrm{sgn}(v(p))$. Then there is a connected open subset F of the level surface of u through p and a warping function f such that a neighborhood $\mathcal{V}_p$ of p in M is isometric to $F_f \times I$. Furthermore, one of the following holds*:

- *$Q_\varphi < 0$ and $p \in \mathcal{V}_- = M$. Then F is a Riemannian manifold of constant positive scalar curvature carrying a non-trivial real Killing spinor and $f(t) = c\cosh(t + d)$.*
- *$Q_\varphi = 0$ and $p \in \mathcal{V}_- = M \backslash u^{-1}(0)$. Then F is a Riemannian manifold of scalar curvature zero carrying a non-trivial parallel spinor and $f(t) = c\,\mathrm{e}^{\pm t}$.*
- *$Q_\varphi > 0$ and $p \in \mathcal{V}_-$. Then F is a Riemannian manifold of constant negative scalar curvature carrying a non-trivial imaginary Killing spinor and $f(t) = c\sinh(t + d)$.*
- *$Q_\varphi > 0$ and $p \in \mathcal{V}_+$. Then F is a Lorentzian manifold of constant positive scalar curvature carrying a non-trivial real Killing spinor.*

Conversely, using the theorems of Section 2, one can see that any warped product $F_f \times I$ with F and f of the types listed above admits a non-trivial real Killing spinor.

Proof. The preceding lemma already assures that there is a neighborhood $\mathcal{V}_p$ of p isometric to such a warped product with warping function $f(t) = u'(t)/u'(0)$ (u being the spinor length along the integral curve of ξ passing through p). Theorems 6 and 7 yield that the restriction of φ to F again induces a non-trivial solution of the Killing equation to Killing number $\pm\lambda_F$.

Furthermore, both theorems together with Theorem 1 imply

$$\frac{R^F}{(n-1)(n-2)} = 4\lambda_F^2 = f^2 + \epsilon(f')^2. \tag{4}$$

Using the definition of f and $u'(t) = \epsilon\sqrt{\epsilon v(t)}$ (see (3)), we get

$$(f'(0))^2 = \left(\frac{u''(0)}{u'(0)}\right)^2 = \left(\frac{-\epsilon u(0)}{u'(0)}\right)^2 = \frac{u^2(0)}{\epsilon v(0)} = \epsilon\frac{u^2(0)}{Q_\varphi - u^2(0)}. \tag{5}$$

Plugging this and $f(0) = 1$ into Eq. (4) gives

$$\frac{R^F}{(n-1)(n-2)} = (f(0))^2 + \epsilon(f'(0))^2 = \frac{Q_\varphi}{Q_\varphi - u^2(0)}. \tag{6}$$

Now $v(p) = Q_\varphi - u^2(0)$ gives the link between R^F and Q_φ. In the case of $p \in \mathcal{V}_-$, the warping function of the local warped product structure is $f(t) = \cosh(t) + (u(0)/u'(0))\sinh(t)$. Now $p \in \mathcal{V}_-$ implies $\epsilon = g(\xi, \xi) = -1$ on $\mathcal{V}_p$. Thus Eq. (5) yields

$$\left(\frac{u(0)}{u'(0)}\right)^2 = \frac{u^2(0)}{u^2(0) - Q_\varphi},$$

and therefore $Q_\varphi < 0$ if and only if $|u(0)/u'(0)| < 1$, $Q_\varphi = 0$ if and only if $|u(0)/u'(0)| = 1$ and finally $Q_\varphi > 0$ if $|u(0)/u'(0)| > 1$. This asserts that f is of the given forms. □

An important application of the preceding theorem is the following.

Theorem 23. *Let $M^{n,1}$ be a connected Lorentzian spin manifold with non-trivial real Killing spinor $\varphi \in \Gamma(S)$. Then M is an Einstein space.*

If $n \leq 4$, then M is of constant sectional curvature.

Proof. The theorem holds for $n = 2$, since a two-dimensional manifold is Einstein if and only if it is of constant scalar curvature. We prove the statement by induction over the dimension of M. Assume the theorem to be true in dimension $n - 1$ and let M be of dimension n. By rescaling, we can assume the Killing number to be $\lambda = \pm\frac{1}{2}$. Theorem 22 implies that a neighborhood $\mathcal{V}_p$ of every point $p \in \mathcal{V}_\pm$ is isometric to a warped product ${F^{n-1}}_f \times (-\delta, \delta)$ and that on F again exists a non-trivial solution of the Killing equation. In the case of Lorentzian F, this solution again is a real Killing spinor and by induction F is an Einstein space. If F is a Riemannian submanifold, then F is Einstein, because every Riemannian spin manifold admitting non-trivial solutions of the Killing equation is Einstein. Now $f'' = -\epsilon f$ and (4) implies

$$R^F = \epsilon(n-1)(n-2)((f')^2 - ff''), \tag{*}$$

and $\mathcal{V}_p$ is Einstein by Lemma 5. Hence, M is Einstein in a neighborhood of every point $p \in \mathcal{V}_\pm$. By continuity and the fact that $\mathcal{V}_+ \cup \mathcal{V}_-$ is dense, M is Einstein space.

The statement for $n \leq 4$ is evident in the case of $n \in \{2, 3\}$, as in those dimensions being Einstein and being of constant sectional curvature is equivalent. For $n = 4$, it is a direct consequence of Lemma 5 and Eq. (*) (again using the fact that in dimension 3 Einstein and constant sectional curvature is the same). □

5.2. *Completeness of warped products*

If F is a Riemannian manifold and $\epsilon = 1$, then for an arbitrary warping function f, one knows that $M = F_f \times I$ is geodesically complete if and only if F is geodesically complete and $I = \mathbb{R}$. This can be proved using metric completeness arguments, which are failing in the case of pseudo-Riemannian manifolds (see [14, Chapter 7, Lemma 40]). A proof of the following lemma can be found in [14, Chapter 7, Proposition 38].

Lemma 24 (Geodesics on warped products). *Let $M = F_f \times I$ be a semi-Riemannian warped product with metric $g_M = f^2(t)g_F + \epsilon\, \mathrm{d}t^2$, $\xi = \partial/\partial t$.*

1. *Then a curve $\gamma = (\beta, \alpha)$ in M is geodesic if and only if, we have*

$$\ddot{\alpha}(s) = \epsilon g(\beta'(s), \beta'(s)) f(\alpha(s)) f'(\alpha(s))\xi \quad \text{in } I,$$
$$\ddot{\beta}(s) = \frac{-2}{f(\alpha(s))}\frac{\mathrm{d}(f\circ\alpha)}{\mathrm{d}s}\beta'(s) \quad \text{in } F \tag{7}$$

   ($\ddot{}$ denoting the second covariant derivative along curves in the factors of the product).
2. *Furthermore, the projection of a geodesic in M on a fiber $F \times \{t\}$ is a pregeodesic.*

Now, we apply this lemma to those Lorentzian warped products which (by Theorem 22) are models for all spaces admitting real Killing spinors.

Corollary 25. *Let $M = F_f \times I$ satisfy one of the following*:

1. *F is Riemannian manifold, $I = \mathbb{R}$, $f(t) = \cosh(t)$ and $\epsilon = -1$;*
2. *F is Riemannian manifold, $I = \mathbb{R}$, $f(t) = \mathrm{e}^t$ and $\epsilon = -1$;*
3. *F is Riemannian manifold, $I = \mathbb{R}_+$, $f(t) = c\sinh(t)$ and $\epsilon = -1$;*
4. *F is Lorentzian manifold, $I = (-(\pi/2), \pi/2)$, $f(t) = \cos(t)$ and $\epsilon = 1$.*

If F is geodesically complete, then M is geodesically complete in the case of (1) *and not geodesically complete in the case of* (2)–(4).

5.3. Global structure of complete Lorentzian manifolds admitting real Killing spinors

In this section, we examine the global structure of the foliation defined by the length function u of a real Killing spinor $\varphi \in \Gamma(S)$ to $\lambda = \pm\frac{1}{2}$ on a connected geodesically complete Lorentzian spin manifold $M^{n,1}$. As above, denote by ξ the normalized gradient field of the length function u of φ defined on $\mathcal{V}_{\pm}$.

Lemma 26. *Let $M^{n,1}$ be geodesically complete and let φ be a real Killing spinor, then the length function $u : M \to \mathbb{R}$ of φ is surjective. The first integral Q_φ is the maximal value of the associated vector field's length function v* (*attained on the level surface $u^{-1}(0)$*).

Proof. Let $p \in M$ arbitrary. Then we find an isotropic vector $X \in T_pM$ such that we have $g(X, V_\varphi(p)) \neq 0$. Let γ be the geodesic defined by $\gamma'(0) = X$. Then along the geodesic γ, u is of the form $u(t) = u(p) \mp g(X, V_\varphi(p))t$ given in Lemma 19 because by Eq. (2), we have $u'(0) = g(\mathrm{grad}(u)_p, X) = \mp g(V_\varphi(p), X)$. This yields surjectivity of u. The second part is a consequence of $Q_\varphi = u^2 + v$. □

Thus, we have for a geodesically complete manifold $M^{n,1}$ with real Killing spinor $\varphi \in \Gamma(S)$:

- $Q_\varphi < 0 \Leftrightarrow M = \mathcal{V}_-$,
- $Q_\varphi = 0 \Leftrightarrow M = \mathcal{V}_- \dot\cup v^{-1}(0)$ (both sets being non-empty),
- $Q_\varphi > 0 \Leftrightarrow M = \mathcal{V}_- \dot\cup v^{-1}(0) \dot\cup \mathcal{V}_+$ (none of the sets being empty),

and $v^{-1}(0) = u^{-1}(\pm\sqrt{Q_\varphi})$.

Theorem 27 (Real Killing spinors with $Q_\varphi < 0$). *Let $\varphi \in \Gamma(S)$ be a real Killing spinor to $\lambda = \pm\frac{1}{2}$ on a geodesically complete manifold $M^{n,1}$. If $Q_\varphi < 0$, then*

$$M \cong F_{\cosh} \times \mathbb{R}^{1,1},$$

where $F = u^{-1}(0)$ is a complete connected Riemannian spin manifold with real Killing spinors. Furthermore,

$$\dim \mathcal{K}_{\pm(1/2)}(M, g_M) = \dim \mathcal{K}_{+(1/2)}(F, g_F) + \dim \mathcal{K}_{-(1/2)}(F, g_F) \quad \text{if } n \text{ is even},$$
$$\dim \mathcal{K}_{\pm(1/2)}(M, g_M) = \dim \mathcal{K}_{+(1/2)}(F, g_F) = \dim \mathcal{K}_{-(1/2)}(F, g_F) \quad \text{if } n \text{ is odd}.$$

Proof. In the case $Q_\varphi < 0$, the vector field ξ is a globally defined geodesic vector field. So its flow exists globally. The map

$$\Psi : F \times \mathbb{R} \to M, \qquad (x, t) \mapsto \Phi_t(x)$$

defined in Lemma 21 is injective for any connected component F of a level surface of u, because u is strictly decreasing along integral curves of ξ (Eq. (3)). As Ψ is a local diffeomorphism, it is a diffeomorphism on its open image.

For any two connected components F and F' of level surfaces of u, the images of the corresponding diffeomorphisms Ψ_F and $\Psi_{F'}$ are equal or disjoint. So by connectivity of M, for any such F, the image of Ψ_F is M. Hence, the level surfaces of u are connected and M is globally a warped product of the form of Lemma 21.

If we take $F = u^{-1}(0)$, then $u(t) = u'(0)\sinh(t)$. So the warping function is of the form $f(t) = u'(t)/u'(0) = \cosh(t)$. F is a totally geodesic submanifold, because the two fundamental forms of F satisfies $\mathbb{I}^F(X, Y) = -(u(0)/\sqrt{\epsilon v(0)}) g(X, Y)\xi = 0$ (that can be seen using the formula $\nabla^M_{\tilde X}\tilde Y = (1/f)\widetilde{\nabla^F_X Y} - \epsilon(f'/f) g_F(X, Y)\xi$, cf. [14]). So geodesic completeness of M and the fact that F is closed in M, implies completeness of F.

The spin geometric part of the statement is a direct consequence of Theorems 6 and 7. □

Theorem 28 (Real Killing spinors with $Q_\varphi = 0$). *Let $M^{n,1}$ be a geodesically complete connected Lorentzian spin manifold with real Killing spinor $\varphi \in \Gamma(S)$ to $\lambda = \pm\frac{1}{2}$. If $Q_\varphi = 0$, then the open sets $\mathcal{V}_-^{\gtrless} = \{p : u(p) \gtrless 0\} \subseteq \mathcal{V}_-$ are isometric to the warped products*

$$F_{e^{\mp t}} \times \mathbb{R}^{1,1},$$

where $F = u^{-1}(\pm 1)$ is a complete Riemannian manifold with parallel spinors. Furthermore,

$$\dim \mathcal{K}_{\pm(1/2)}(\mathcal{V}_-^{\gtrless}, g_M) \geq \dim \mathcal{K}_0(F, g_F) \quad \textit{if } n \textit{ is even},$$

and

$$\dim \mathcal{K}_{\mp\kappa}(\mathcal{V}_-^{\gtrless}, g_M) \geq \dim \mathcal{K}_0^-(F, g_F),$$
$$\dim \mathcal{K}_{\pm\kappa}(\mathcal{V}_-^{\gtrless}, g_M) \geq \dim \mathcal{K}_0^+(F, g_F) \quad \textit{if } n \textit{ is odd} \ \left(\textit{where } \kappa = \tfrac{1}{2}(-1)^m\right).$$

Proof. Let $p \in \mathcal{V}_-^{\gtrless} = \{p : u(p) \gtrless 0\} \subseteq \mathcal{V}_-$. Denote γ the geodesic with $\gamma'(0) = \xi(p)$. By Eq. (6) and $u'(0) < 0$, we have $u(0)/u'(0) = \mp 1$, so the length function u along with γ satisfies

$$u(t) = u(0)\cosh(t) + u'(0)\sinh(t) = u(0)\,\mathrm{e}^{\mp t}.$$

Hence, for any $p \in \mathcal{V}_-^{\gtrless}$, the geodesic γ stays in $\mathcal{V}_-^{\gtrless}$ and runs through all level surfaces of u lying in $\mathcal{V}_-^{\gtrless}$.

If we take $F = u^{-1}(\pm 1)$, the map

$$\Psi : F \times \mathbb{R} \to \mathcal{V}_-^{\gtrless}, \qquad (x, t) \mapsto \Phi_t(x)$$

defined in Lemma 21 is a diffeomorphism. It is injective, because u is strictly decreasing along geodesic (Eq. (3)), and it is surjective, because for any point in $\mathcal{V}_-^{\lessgtr}$, the geodesic γ, which is the integral curve of ξ runs through all level surfaces of u lying in $\mathcal{V}_-^{\gtrless}$.

Therefore, by Lemma 21 the sets $\mathcal{V}_-^{\lessgtr}$ are isomorphic to warped products, the warping function being $f(t) = \mathrm{e}^{\mp t}$ by the above formula for u along integral curves of ξ.

The spin geometric part of the statement follows from Theorems 6 and 7. Showing that F is complete is a little bit more technical as in the preceding case. We prove that there is an $\epsilon > 0$ such that for any $p \in F$ and any $X \in T_pF$ of length 1, the geodesic in F being tangent to X in p exists at least on the interval $(-\epsilon, \epsilon)$. As ϵ is independent of p and X, every geodesic in F can be extended to $\mathbb{R}$.

Let γ be the geodesic in M with $\gamma'(0) = X \in T_pF$. Because γ is spacelike and has length 1, by Lemma 19 we have $u(\gamma(s)) = \pm\cos(s)$. Thus, for $s \in (-(\pi/2), \pi/2)$, γ stays in $\mathcal{V}_-^{\lessgtr}$. There we have $\gamma = (\beta, \alpha) \in F \times \mathbb{R}$. By Lemma 24, β is pregeodesic in F with $\beta'(0) = X$. Parameterizing β on arc-length yields a geodesic $\tilde{\beta}$. What we have to show is that the arc-length of β in both directions from 0 is at least ϵ, where ϵ is independent of p and X. Then F is geodesically complete, because every geodesic can be extended.

Because u is constant on the fibers of the warped product we have $u(\gamma(s)) = u((p, \alpha(s)))$ which is equivalent to $\pm\cos(s) = \pm\mathrm{e}^{\mp\alpha(s)}$. Thus, independent of p and X, we have that $\alpha(s) = \mp\ln(\cos(s))$. By Lemma 24, $l(s) = g(\beta'(s), \beta'(s))$ satisfies

$$l'(s) = 2g_F(\ddot{\beta}(s), \beta'(s)) = -\frac{4}{f \circ \alpha}\frac{\mathrm{d}(f \circ \alpha)}{\mathrm{d}s} l(s),$$

and independent of p and X

$$l(s) = \frac{1}{f(\alpha(s))^4} = \frac{1}{e^{\mp(\mp \ln(\cos(s)))}} = \frac{1}{\cos^4(s)}.$$

If one takes $\epsilon = \int_0^{\pi/4}(1/\cos^2(s))\,ds$ and $L(s) = \int_0^s \sqrt{l(\tau)}\,d\tau = \int_0^s (1/\cos^2(\tau))\,d\tau$, so

$$\tilde{\beta} : (-\epsilon, \epsilon) \xrightarrow{L^{-1}} \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \xrightarrow{\beta} F$$

is a geodesic. □

Real Killing spinors with $Q_\varphi > 0$. Let $M^{n,1}$ be a geodesically complete connected Lorentzian spin manifold with non-trivial real Killing spinor $\varphi \in \Gamma(S)$ to $\pm\frac{1}{2}$ and $Q_\varphi > 0$.

In this case, the level surfaces $u^{-1}(\pm\sqrt{Q_\varphi})$ are degenerated and decompose M into the following three open sets (given curvatures following form Eq. (6)):

- $\mathcal{V}_-^> = \{p : u(p) > +\sqrt{Q_\varphi}\}$ containing level surfaces of u being Riemannian submanifolds with negative scalar curvature and timelike normal vector field ξ,
- $\mathcal{V}_+ = \{p : u^2(p) < Q_\varphi\}$ containing level surfaces of u being Lorentzian submanifolds with positive scalar curvature and spacelike normal vector field ξ,
- $\mathcal{V}_-^< = \{p : u(p) < -\sqrt{Q_\varphi}\}$ containing level surfaces of u being Riemannian submanifolds with negative scalar curvature and timelike normal vector field ξ.

Theorem 29 (Real Killing spinors with $Q_\varphi > 0$ on $\mathcal{V}_+$). *Let $M^{n,1}$ be a geodesically complete Lorentzian manifold with real Killing spinor $\varphi \in \Gamma(S)$ to $\pm\frac{1}{2}$ and $Q_\varphi > 0$. Then $\mathcal{V}_+$ is isometric to*

$$F_{\cos(t)} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

where $F = u^{-1}(0)$ is a geodesically complete Lorentzian manifold admitting real Killing spinors. Furthermore,

$$\dim \mathcal{K}_{\pm(1/2)}(\mathcal{V}_+, g_M) = \dim \mathcal{K}_{+(1/2)}(F, g_F) + \dim \mathcal{K}_{-(1/2)}(F, g_F) \quad \text{if } n \text{ is even},$$
$$\dim \mathcal{K}_{\pm(1/2)}(\mathcal{V}_+, g_M) = \dim \mathcal{K}_{+(1/2)}(F, g_F) = \dim \mathcal{K}_{-(1/2)}(F, g_F) \quad \text{if } n \text{ is odd}.$$

Proof. Let $F = u^{-1}(0)$, then the length function u along the geodesic γ with $\gamma'(0) = \xi(x)$ satisfies $u(\pm(\pi/2)) = \pm\sqrt{Q_\varphi}$ (because Lemma 19 and Eq. (3) imply $u(t) = \sqrt{Q_\varphi}\sin(t)$) for every $p \in F$. The map

$$\Psi : F \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathcal{V}_+, \qquad (x, t) \mapsto \Phi_t(x)$$

is injective because by Eq. (3), u is strictly increasing along integral curves of ξ and therefore diffeomorphism on its image.

So, we only have to show that Ψ is surjective. Let $p \in \mathcal{V}_+$ arbitrary and γ be the geodesic tangent to ξ in p. Then γ stays in $\mathcal{V}_+$ on an interval (t_-, t_+) containing p. The boundaries of the interval are defined to be the numbers closest to zero such that $u(t_\pm) = \pm\sqrt{Q_\varphi}$. So every point is connected to F by an integral curve of ξ and Ψ is surjective.

By Lemma 21, Ψ is an isometry for the warping function $f(t) = u'(t)/u'(0) = \cos(t)$. F is a totally geodesic submanifold, because for any two vector fields $X, Y \in \mathfrak{X}(F)$, we have

$$\mathbb{I}^F(X,Y) = \mathrm{proj}_\xi(\nabla^M_X Y) = -g(\nabla^M_X Y, \xi)\xi = g(Y, \nabla^M_X \xi) = -\frac{u}{\sqrt{\epsilon v}} g(X,Y) = 0,$$

where $\mathbb{I}^F$ is the two fundamental forms of F. So geodesic completeness of M and the fact that F is closed in M, implies geodesic completeness of F.

The spin geometric part of the statement is a direct consequence of Theorems 6 and 7. □

Theorem 30 (Real Killing spinors with $Q_\varphi > 0$ on $\mathcal{V}_-$). *Let $M^{n,1}$ be a geodesically complete Lorentzian spin manifold with real Killing spinor $\varphi \in \Gamma(S)$ to $\lambda = \pm\frac{1}{2}$ with $Q_\varphi > 0$. Then the open sets $\mathcal{V}_-^{\gtrless} = \{p : u(p) \gtrless \pm\sqrt{Q_\varphi}\}$ are isometric to*

$$F_f \times I^{1,1},$$

where F is a level surface $u = \pm c$ (for arbitrary $c > \sqrt{Q_\varphi}$) and

$$f(t) = \cosh(t) \mp \sqrt{\frac{c^2}{c^2 - Q_\varphi}}\sinh(t),$$

and

$$I = \begin{cases} \left(-\infty, \quad \mathrm{Artanh}\left(+\sqrt{\frac{c^2-Q_\varphi}{c^2}}\right)\right), & u = +c, \\ \left(\mathrm{Artanh}\left(-\sqrt{\frac{c^2-Q_\varphi}{c^2}}\right), \quad \infty\right), & u = -c. \end{cases}$$

F is a complete Riemannian spin manifold with

$$\begin{aligned} \dim \mathcal{K}_{\pm(1/2)}(\mathcal{V}_-^{\gtrless}, g_M) &= \dim \mathcal{K}_{(\mathrm{i}/2)\sqrt{Q_\varphi/(c^2-Q_\varphi)}}(F, g_F) \\ &\quad + \dim \mathcal{K}_{-(\mathrm{i}/2)\sqrt{Q_\varphi/(c^2-Q_\varphi)}}(F, g_F) \quad \textit{for } n \textit{ even}, \\ \dim \mathcal{K}_{\pm(1/2)}(\mathcal{V}_-^{\gtrless}, g_M) &= \dim \mathcal{K}_{(\mathrm{i}/2)\sqrt{Q_\varphi/(c^2-Q_\varphi)}}(F, g_F) \\ &= \dim \mathcal{K}_{-(\mathrm{i}/2)\sqrt{Q_\varphi/(c^2-Q_\varphi)}}(F, g_F) \quad \textit{for } n \textit{ odd}. \end{aligned}$$

Proof. Let $p \in \mathcal{V}_-^{\gtrless}$. Let γ be the geodesic such that $\gamma'(0) = \xi(0)$. The length function $u(t) = u(0)\cosh(t) + u'(0)\sinh(t)$ along γ is strictly decreasing, as long as γ stays in $\mathcal{V}_-$. Let $t_0 = \mathrm{Artanh}(-(u'(0)/u(0))) \overset{(5)}{=} \mathrm{Artanh}(\pm\sqrt{(u^2(0) - Q_\varphi)/u^2(0)})$ be the zero of $f(t) = u'(t)/u'(0)$ and extremal point of u.

We have $u(t_0) = \mathrm{sgn}(u(p))\sqrt{Q_\varphi}$, because

$$
\begin{aligned}
u(t_0) &= \cosh(t_0)\left(u(0) - u'(0)\tanh\left(\mathrm{Artanh}\left(-\frac{u'(0)}{u(0)}\right)\right)\right) \\
&= \cosh(t_0)\left(\frac{u^2(0) - (u')^2(0)}{u(0)}\right) \overset{(3)}{=} \cosh(t_0)\left(\frac{u^2(0) + v(0)}{u(0)}\right) \\
&= \cosh(t_0)\left(\frac{Q_\varphi}{u(0)}\right),
\end{aligned}
$$

and $\cosh^2(t_0) = 1/(1 - \tanh^2(t_0)) = u^2(0)/Q_\varphi$ (by definition of t_0).

At time t_0, the geodesic γ intersects one of the degenerate level surfaces given by $v^{-1}(0) = u^{-1}(\sqrt{\pm Q_\varphi})$.

For $c > \sqrt{Q_\varphi}$ and $F = u^{-1}(\pm c)$. Then

$$
\Psi : F \times I \to \mathcal{V}_{-}^{\gtrless}, \qquad (x, t) \mapsto \Phi_t(x)
$$

is defined for

$$
I = \begin{cases} \left(-\infty,\quad \mathrm{Artanh}\left(+\sqrt{\dfrac{c^2 - Q_\varphi}{c^2}}\right)\right), & u = +c, \\[2ex] \left(\mathrm{Artanh}\left(-\sqrt{\dfrac{c^2 - Q_\varphi}{c^2}}\right),\quad \infty\right), & u = -c. \end{cases}
$$

The map is injective and local diffeomorphism. It is surjective, because for any $p \in \mathcal{V}_{-}^{\gtrless}$ the geodesic γ through ξ in p intersects F.

The spin geometric part of the statement is a direct consequence of Theorems 6 and 7. F can be proved to be geodesically complete by the same method as in the case $Q_\varphi = 0$. □

5.4. *Real Killing spinors with $Q_\varphi < 0$*

In the preceding section, we have proved that any complete Lorentzian spin manifold of dimension $n \geq 3$ admitting a non-trivial real Killing spinor to $\pm\frac{1}{2}$ with first integral $Q_\varphi < 0$ is a warped product of the form

$$
M^{n,1} = F^{n-1}{}_{\cosh} \times \mathbb{R}^{1,1},
$$

where F is a complete Riemannian manifold admitting a real Killing spinor. Conversely, Theorems 6 and 7 imply that every manifold of this form admits a real Killing spinor. The question is, whether it admits a Killing spinor φ with first integral $Q_\varphi < 0$.

From Theorem 4, we immediately get the following list of all possible candidates for complete simply connected Lorentzian manifolds with Killing spinors to $\pm\frac{1}{2}$ with $Q_\varphi < 0$:

- $\mathbb{S}^{n,1} = \mathbb{S}^{n-1}{}_{\cosh} \times \mathbb{R}^{1,1}$ for all $n \geq 3$,

- $M^{2m+2,1} = F^{2m+1}{}_{\cosh} \times \mathbb{R}^{1,1}$, where F is a complete 1-connected Einstein–Sasaki manifold ($m \geq 2$),
- $M^{7,1} = F^6{}_{\cosh} \times \mathbb{R}^{1,1}$, where F^6 is a complete 1-connected nearly Kähler non-Kähler spin manifold, and
- $M^{8,1} = F^7{}_{\cosh} \times \mathbb{R}^{1,1}$, where F^7 is one of the complete 1-connected manifolds admitting a "nice" 3-form.

The next two theorems show that at least in the first two cases there is a Killing spinor with $Q_\varphi < 0$ (except on the sphere $\mathbb{S}^{3,1}$, where $Q_\varphi = 0$ for all spinors). The proof of both theorems involve explicit calculation with the spinor representation that we do not want to carry out here (see [9] for details).

Theorem 31. *Let F^{2m+1} be a complete 1-connected Einstein–Sasaki manifold ($m \geq 2$), then*

$$M^{2m+2,1} = F^{2m+1}{}_{\cosh} \times \mathbb{R}^{1,1}$$

is a complete 1-connected Lorentzian spin manifold admitting a real Killing spinor φ to $\pm\frac{1}{2}$ with first integral $Q_\varphi < 0$.

Theorem 32. *On the Lorentzian spheres $\mathbb{S}^{n,1} = \mathbb{S}^{n-1}{}_{\cosh} \times \mathbb{R}^{1,1}$ of dimension $n \geq 4$, there exists a Killing spinor $\varphi \in \Gamma(S)$ with $Q_\varphi < 0$ for each Killing number $\pm\frac{1}{2}$.*

In both cases, we even show that if we have a warped product of the given form, then there is a Killing spinor φ which recovers the original warped product structure, i.e. the normed gradient field ξ of the length function $u = \langle\varphi, \varphi\rangle$ of the spinor is equal to $\partial/\partial t$. In the case that F is an Einstein–Sasaki manifold which is neither a sphere nor a Sasaki-3 manifold, all Killing spinors φ satisfy $Q_\varphi < 0$ and recover the warped product structure.

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